# ON PROPAGATION OF SHOCK WAVES IN AN ELASTO-PLASTIC MEDIUM

PMM Vol. 33, No. 1, 1969, pp. 143-147
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(Received February 1, 1968)

Investigations of shock waves are well known in gases [1,2], in elastic media [3] and many others. These investigations pertain to the case where it is possible to obtain directly a closed system of equations at the discontinuities. For the majority of rheological media it is impossible to write directly the determining equations at the discontinuities. This situation arises because it is necessary on the shock wave to know the character of change of some discontinuous quantity $f$ in its dependence on another discontinuous quantity $\varphi$. In this work it is proposed to find the dependence $f(\varphi)$ from the analysis of the shock transition.

1. Let us assume that the surface of discontinuity can be replaced by a thin transition layer of thickness $2h$. Let us use $x_3$ to denote the Cartesian coordinate which has a direction normal to some mean surface $\Sigma$ which is located within the transition layer. Let the surface $\Sigma$ coincide with the surface of a strong discontinuity such that the transition layer reduces to this surface for $h \to 0$.

Smooth changes in the transition layer before the "discontinuous" quantities on the shock waves cannot be found from the solution of the problem of the structure of the shock wave in the medium under study. After solution of this problem the dependence of "discontinuous" quantities on the coordinate $x_3$ will be found in the form

$$f = f(x_3), \qquad \varphi = \varphi(x_3) \tag{1.1}$$

Functions $f$ and $\varphi$ in (1.1) may depend on time $t$ and three spatial coordinates $x_i$, where $x_1$ and $x_2$ will play the part of parameters. Eliminating $x_3$ from the first relationship in (1.1) with the aid of the second relationship, we find the desired relationship $f(\varphi)$ which is valid inside the shock layer.

The structure of the transition layer depends on the unknown velocity of propagation of the shock wave $G$. In this manner we arrive at the necessity of simultaneous solution of of two problems: the problem of the structure of the shock wave and the problem of the propagation of the shock wave in the given continuous medium. The structure of the shock layer may be described through dissipation processes of viscosity. If the coefficients of viscosity are allowed to approach zero, the thickness of the transition layer $2h$ also tends to zero. In the limit the smooth variation of values transforms into a discontinuous one, while the relationship $f(\varphi)$ tends to some limiting expression of its own.

The complex problem of structure and propagation of the shock wave may turn out to be very difficult. In connection with this an approximate method for the solution of the problem is proposed.

The relationship $f(\varphi)$ is presented in the form of a sum

$$f(\varphi) = f^+ + \frac{\varphi - \varphi^+}{[\varphi]}[f] + \psi(\varphi) \tag{1.2}$$

The plus sign above quantities in (1.2) designates that this quantity is calculated on the forward shock front. The square brackets designate the discontinuity. If the relationship (1.2) in the space of variables $(f, \varphi)$ is approximated by a straight line which passes through two points $(f^+, \varphi^+)$ and $(f^-, \varphi^-)$, then the term $\psi$ in (1.2) should be neglected.

It follows from (1.2) that

$$\psi^+ = \psi(\varphi^+) = 0, \qquad \psi^- = \psi(\varphi^-) = 0 \tag{1.3}$$

In relationships (1.2) we shall subsequently neglect quantities $\psi$, which satisfy conditions (1.3). Such an approximation is valid for shock waves of small intensity, when points $(f^+, \varphi^+)$ and $(f^-, \varphi^-)$ in the space $(f, \varphi)$ are located close to one another.

Let the flow outside the transition layer of the elasto-plastic medium be described by a system of determining equations of Prandtl-Reuss [3] with the plasticity condition of Mises [4]. The existence of the transition layer is caused by the appearance of additional properties of the medium within the shock layer. The rheological model of the elasto-plastic medium within the transition layer is represented in Fig. 1. Additional properties of the medium are determined by the structure of element $D$. Usually this element will be the element of viscosity [5,6].

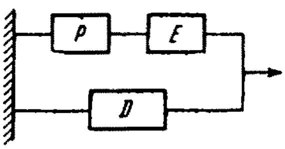


Fig. 1

The plasticity conditions of Mises and the determining equations for such a medium are written in the form

$$(\sigma_{ij}^* - d_{ij}^*)(\sigma_{ij} - d_{ij}) = 2k^2 \tag{1.4}$$

$$\frac{D(\sigma_{ij} - d_{ij})}{Dt} = \lambda v_{k,k}\delta_{ij} + \mu\left\{v_{i,j} + v_{j,i} - 2\frac{d\varphi}{dt}(\sigma_{ij}^* - d_{ij}^*)\right\} \tag{1.5}$$

Here $\sigma_{ij}$ is the stress tensor in the medium; $d_{ij}$ is the stress tensor in the rheological element $D$; $v_i$ is the velocity of material particles; $k$ is the limit of fluidity; $\lambda$ and $\mu$ are Lame elasticity constants. The asterisk above the tensor indicates the deviator part of these tensors, $D/Dt$ is the covariant derivative with respect to time in the sense of Jaumann [7]

$$\frac{D\sigma_{ij}}{Dt} = \frac{d\sigma_{ij}}{dt} + \frac{1}{2}\sigma_{ik}(v_{k,j} - v_{j,k}) + \frac{1}{2}\sigma_{jk}(v_{k,i} - v_{i,k}) \tag{1.6}$$

It is assumed that the dissipative properties of the medium which are described by the rheological element $D$ are apparent only within the transition layer. Approaching the forward or rear shock front these properties gradually disappear so that the following equations may be assumed to be valid:

$$d_{ij}^+ = d_{ij}^- = 0 \tag{1.7}$$

Let us introduce a moving system of rectilinear coordinates such that its origin moves together with the surface $\Sigma$. At an arbitrary mass point under examination on the surface $\Sigma$ let us orient the axis $x_3$ along the normal to this surface, then the axes $x_1$ and $x_2$ will be located in the tangential plane. Let the Greek indices $\alpha, \beta, \ldots$ assume the values 1 or 2, and the Latin indices $i, j, k, \ldots$ the values 1, 2 or 3. All quantities will be calculated in the stationary system of coordinates and will be projected on the axes of the moving system.

In (1.5) it is necessary to separate derivatives with respect to the normal to the surface of discontinuity from derivatives with respect to the tangential directions. The material derivative with respect to time is replaced by a $\delta$-derivative. For this the following relationships are required:

$$\frac{\partial}{\partial x_i} = \delta_{i3}\frac{\partial}{\partial x_3} + \delta_{i\alpha}\frac{\partial}{\partial x_\alpha}, \qquad \frac{\partial}{\partial t} = \frac{\delta}{\delta t} - G\frac{\partial}{\partial x_3} \tag{1.8}$$

Let us write the dynamic conditions for discontinuities of density $\rho$, velocity and stresses

$$[\sigma_{i3}] = \rho^+ (v_3^+ - G) [v_i], \qquad [\rho (v_3 - G)] = 0 \tag{1.9}$$

In order to obtain a closed system of eleven equations with respect to jumps of quantities $\rho, \varphi, v_i$ and $\sigma_{ij}$, we shall integrate the determining equations (1.5) across the transition layer. After this we find in the limit $h \to 0$

$$\int_{\sigma_{ij}^-}^{\sigma_{ij}^+} (v_3 - G)\, d(\sigma_{ij} - d_{ij}) + \frac{1}{2} \int_{v_\alpha^-}^{v_\alpha^+} \{(\sigma_{i\alpha} - d_{i\alpha}) \delta_{j3} + (\sigma_{j\alpha} - d_{j\alpha}) \delta_{i3} -$$

$$- (\sigma_{i3} - d_{i3}) \delta_{j\alpha} - (\sigma_{j3} - d_{j3}) \delta_{i\alpha}\}\, dv_\alpha =$$

$$= \lambda [v_3] \delta_{ij} + \mu [v_i \delta_{j3} + v_j \delta_{i3}] - 2\mu \int_{\varphi^-}^{\varphi^+} (\sigma_{ij}^* - d_{ij}^*) (v_3 - G) d\varphi \tag{1.10}$$

According to (1.2), (1.3) and (1.7) it is appropriate to neglect the quantities $d_{ij}$ in (1.10) in the present approximation in view of their smallness in comparison to other terms. Taking advantage of this situation and the linear approximation for the dependence between discontinuity quantities (1.2), after integration, Eq. (1.10) will have the form

$$[{}^1/_2 (v_3^+ + v_3^-) - G] [\sigma_{ij}] + {}^1/_4 \{(\sigma_{i\alpha}^+ + \sigma_{i\alpha}^-) \delta_{j3} + (\sigma_{j\alpha}^+ + \sigma_{j\alpha}^-) \delta_{i3} - (\sigma_{i3}^+ + \sigma_{i3}^-) \delta_{j\alpha} -$$

$$- (\sigma_{j3}^+ + \sigma_{j3}^-) \delta_{i\alpha}\} [v_\alpha] = \lambda [v_3] \delta_{ij} + \mu [v_i \delta_{j3} + v_j \delta_{i3}] -$$

$$- \mu \{\sigma_{ij}^{*+} (v_3^+ - G) + \sigma_{ij}^{*-} (v_3^- - G)\} [\varphi] \tag{1.11}$$

The plasticity condition (1.4) which is written for discontinuities taking into consideration (1.7), will be written in the form

$$(\sigma_{ij}^{*+} + \sigma_{ij}^{*-}) [\sigma_{ij}] = 0 \tag{1.12}$$

The system of eleven nonlinear equations (1.9), (1.11) and (1.12) contains eleven jumps of quantities $\rho, \varphi, v_i, \sigma_{ij}$ and the unknown velocity $G$. In order to find $G$ by simple transformations, we shall reduce this system to three equations which contain from discontinuities only the jumps in velocity $v_i$.

In (1.11) we equate subscripts $i$ and $j$ and sum with respect to the repeating subscript, and then we obtain

$$[\sigma_{kk}] = 2 (3\lambda + 2\mu) [v_3] (v_3^+ + v_3^- - 2G)^{-1} \tag{1.13}$$

After multiplication of (1.11) by $(\sigma_{ij}^{*+} + \sigma_{ij}^{*-})$ and utilizing here (1.4) and (1.12), we arrive at the equation

$$(\sigma_{i3}^{*+} + \sigma_{i3}^{*-}) [v_i] = 2k^2 (v_3^+ + v_3^- - 2G) [\varphi] \tag{1.14}$$

If it is assumed in (1.11) that $i = \alpha$ and $j = \beta$, we shall have

$$[{}^1/_2 (v_3^+ + v_3^-) - G] [\sigma_{\alpha\beta}] - {}^1/_4 (\sigma_{\alpha 3}^+ + \sigma_{\alpha 3}^-) [v_\beta] - {}^1/_4 (\sigma_{\beta 3}^+ + \sigma_{\beta 3}^-) [v_\alpha] = \tag{1.15}$$

$$= \lambda [v_3] \delta_{\alpha\beta} - \mu \{\sigma_{\alpha\beta}^{*+} (v_3^+ - 2G + v_3^-) - (v_3^- - G) [\sigma_{\alpha\beta}] + {}^1/_3 (v_3^- - G) [\sigma_{kk}] \delta_{\alpha\beta}\} [\varphi]$$

For $j = 3$ we find from (1.11)

$$[{}^1/_2 (v_3^+ + v_3^-) - G] [\sigma_{i3}] + \frac{1}{4} \{\sigma_{i\alpha}^+ + \sigma_{i\alpha}^- + (\sigma_{3\alpha}^+ + \sigma_{3\alpha}^-) \delta_{i3} - (\sigma_{33}^+ + \sigma_{33}^-) \delta_{i\alpha}\} [v_\alpha] =$$
$$= \lambda [v_3] \delta_{i3} + \mu [v_i + v_3\delta_{i3}] - \mu \{\sigma_{i3}^{*+} (v_3^+ - G) + \sigma_{i3}^{*-} (v_3^- - G)\} [\varphi] \qquad (1.16)$$

From equations (1.13) - (1.15) it is consequently possible to express jumps $[\sigma_{kk}]$, $[\varphi]$ and $[\sigma_{\alpha\beta}]$ through jumps $[v_i]$, in this connection making use of relationship (1.9). Substituting these jumps into (1.16), we shall obtain a system of three nonlinear equations with respect to jumps $[v_i]$ and velocity $G$. Assuming one of these jumps as given, after elimination of two other velocity jumps from three equations we shall obtain an equation with respect to $G$. From the obtained equation in $G$ it is appropriate to find the velocities of particles of the medium with an accuracy to the first power of the given jump. This follows from the linear approximation in (1.2). Without writing out this system we note that for shock waves of very small intensity, if it can be linearized, it will assume the form

$$\Big\{\rho^+ (v_3^+ - G)^2 \delta_{ij} - (\lambda + \mu) \delta_{i3}\delta_{j3} - \mu\delta_{ij} + \frac{\mu}{k^2} \sigma_{i3}^{*+}\sigma_{j3}^{*+} +$$
$$+ \frac{1}{2} (a_{ij} - a_{i3}\delta_{j3})\Big\} [v_j] = 0, \quad a_{i\alpha} = \sigma_{i\alpha}^+ + \sigma_{3\alpha}\delta_{i3}^+ - \sigma_{33}^+\delta_{i\alpha} \qquad (1.17)$$

The terms $a_{i\alpha}$ take into account the effect of rotation of the surroundings of the material point of the medium. Equating the determinant of the homogenous linear system of equations (1.17) to zero, we obtain a cubic equation for finding $\rho^+ (G - v_3^+)^2$. For the case of an irrotational wave we find from (1.17)

$$\rho^+ (G - v_3^+)^2 = \lambda + 2\mu - \mu \left(\frac{\sigma_{33}^{*+}}{k}\right)^2 \qquad (1.18)$$

If the shock wave is equivoluminous, we shall obtain from (1.17) for the particular case when $[v_3] = 0$

$$\rho^+ (G - v_3^+)^2 = \mu - \mu (\sigma_{13}^+ / k)^2 + {}^1/_2 (\sigma_{33}^+ - \sigma_{11}^+) \qquad (1.19)$$

In the general case when $[v_i] \neq 0$, only positive roots among the solutions of the cubic equations have a significance.

The second law of thermodynamics [8] places a limit on the propagation of shock waves. According to this law the power energy dissipation as a result of plastic flow cannot be negative, i.e. [4]

$$\frac{d\varphi}{dt} \geqslant 0 \qquad (1.20)$$

In order to write inequality (1.20) at the discontinuities, we shall integrate it across the transition layer. The sign of inequality does not change in this connection, because the upper limit of integration $h$ is greater than the lower limit $-h$. In the limit for $h \to 0$ we shall have

$$\lim_{h\to 0} \int_{-h}^{h} \frac{d\varphi}{dt} dx_3 = \int_{\varphi^-}^{\varphi^+} (v_3 - G)\, d\varphi = (v_3^* - G) [\varphi] \geqslant 0$$

$$\max (v_3^+, \; v_3^*) \geqslant v_3^* \geqslant \min (v_3^+, v_3^-) \qquad (1.21)$$

From the law of conservation of mass on the surface of the discontinuity (1.9) it follows that $(G - v_3^+)$ and $(G - v_3^-)$, consequently also $(G - v_3^+)$ have the same signs. In connection with this we obtain from (1.14) and (1.21)

$$(\sigma_{i3}^{*+} + \sigma_{i3}^{*-})\,[v_i] \geqslant 0 \tag{1.22}$$

In this manner the propagation of the shock wave in an elasto-plastic medium is possible from the thermodynamic point of view if the inequality (1.22) is satisfied on the surface of the discontinuity.

**2.** In addition to the general case of propagation of the shock wave which was examined, it is also of interest in the elasto-plastic medium to examine the particular case when on the surface of the discontinuity

$$[\sigma_{33}^*] = 0, \qquad 3\,[\sigma_{33}] = [\sigma_{kk}] \tag{2.1}$$

In this case it is easy to find the velocity $G$. Solving simultaneously (2.1), (1.13) and Eq. (1.9) for $i = 3$, we obtain

$$G - v_3^+ = -{}^1/_4\,[v_3] \pm \{(\lambda + 2\mu/3)\,\rho^+\}^{1/2} \tag{2.2}$$

The assumption (2.1) places a limit on the propagation of the shock wave with velocity (2.2). For a shock wave of very small intensity this limit has the form

$$4k^2 - 3\sigma_{i3}^{*+}\sigma_{j3}^{*+}\omega_i\omega_j = {}^1/_2 a_{i\alpha}\omega_i\omega_\alpha, \qquad [v_i] = \omega\omega_i \tag{2.3}$$

This equation is obtained by multiplying (1.17) by $\omega_i$ with utilization of (2.2). Here $\omega$ is the intensity of the shock wave, $\omega_i$ are the directional cosines of the vector $[v_i]$. Neglecting terms $a_{i\alpha}$, we shall write (2.3) in the invariant form in the stationary system of coordinates

$$2\sigma_{ij}^{*+}\sigma_{ij}^{*+} = 3\,(\sigma_{ij}^{*+}\nu_i\omega_j)^2 \tag{2.4}$$

where $\nu_i$ are the direction cosines of the normals to the surface of the discontinuity.

We shall write the expression (2.4) in a system of coordinates which coincides with the principal directions of the tensor $\sigma_{ij}$

$$2\sigma_k^{*+}\sigma_k^{*+} = 3\,(\sigma_1^{*+}\nu_1\omega_1 + \sigma_2^{*+}\nu_2\omega_2 + \sigma_3^{*+}\nu_3\omega_3)^2 \tag{2.5}$$

Here $\sigma_k$ are the principal values of the stress tensor. We shall find the extremum $z$ of the right side of this equation as a function of directions of vectors $\nu_k$ and $\omega_k$

$$z = \sigma_1^{*+}\nu_1\omega_1 + \sigma_2^{*+}\nu_2\omega_2 + \sigma_3^{*+}\nu_3\omega_3, \qquad \nu_k\nu_k = \omega_k\omega_k = 1 \tag{2.6}$$

In the space of variables $(z, \nu_k, \omega_i)$ the surface (2.6) will be closed and smooth, therefore the greater extremum gives a maximum for $z$. The calculationswill be omitted, but after examination of the maximum for $z$ we find that a solution of Eq. (2.5) with respect to $\nu_k$ and $\omega_k$ is possible only in the case when two out of three principal values $\sigma_k$ coincide, and vectors $\nu_k$ and $\omega_k$ coincide with the third principal direction.

Changing again to the moving system of coordinates this result will have the form

$$\sigma_{11}^* = \sigma_{22}^* = -{}^1/_2\,\sigma_{33}^* = \pm\,{}^1/_3\sqrt{3}\,k, \; \sigma_{13} = \sigma_{23} = \sigma_{12} = 0, \qquad [v_\alpha] = 0 \tag{2.7}$$

The last equation in (2.7) says that the examined shock wave will be irrotational. Conditions (2.7) can be satisfied in the region of one-dimensional flow, when the state of stress on the surface of the discontinuities has spherical symmetry. For the jumps of the plastic part of deformation $[e^{p}_{ij}]$ we find from (1.11), (1.14) and (2.7) the expression

$$[e_{11}^{p}] = [e_{22}^{p}] = -\tfrac{1}{2}[e_{33}^{p}] = \frac{[v_3]}{3(G - v_3^{+})}, \qquad [e_{12}^{p}] = [e_{13}^{p}] = [e_{23}^{p}] = 0 \qquad (2.8)$$

The second law of thermodynamics in the form (1.22) is transformed into the inequality

$$\sigma_{33}^{*}\,[v_3] \geqslant 0 \qquad (2.9)$$

The approximate method offered in this paper allows to write the determining equations for an elasto-plastic medium with an accuracy to the squares of discontinuity quantities. From the obtained system of equations it is possible to determine the velocity of propagation of the shock wave with an accuracy to the first power of the jump of the given quantity. For a more accurate solution of the problem of shock wave propagation it is necessary first to solve the problem of the structure of this wave. It is possible to show by direct calculations that shock waves of very small intensity and waves of weak discontinuity have the same properties with the exception of inequality (1.22). For weak waves of discontinuity it does not apply.

A similar approach can be utilized for the solution of the problem of propagation of shock waves in other complex media.

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Translated by B.D.